DOI: 10.1103/PhysRevD.104.044014

Two-fluid stellar objects in general relativity: The covariant formulation

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Abstract

We apply the 1+1+2 covariant approach to describe a general static and spherically symmetric relativistic stellar object which contains two interacting fluids. We then use the 1+1+2 equations to derive the corresponding Tolman-Oppenheimer-Volkoff equations in covariant form in the isotropic noninteracting case. These equations are used to obtain three new exact solutions by means of direct resolution and reconstruction techniques, as well as a two-fluid generalization of the interior Schwarzschild solution. Finally, we show that one of the generating theorems known for the single-fluid case can also be used to obtain a new exact two-fluid solution from a single-fluid one.

The two-fluid covariant TOV equations

Coordinate invariant and tetrad methods are an important way of transforming the equations of general relativity into first-order ODEs, as opposed to second-order PDEs. The approach is most useful in the presence of homogeneity, isotropy, and spacetimes that admit a high degree of symmetry. The 1+1+2 formalism that we employ can be considered a semi-tetrad approach because it relies on both a time-like and a spacelike threading. We, therefore, start constructing the 1+1+2 formalism from the threading decomposition of the spacetime. In this way, we can construct a set of tensorial objects connected to the properties of the field lines, which make up the set of 1+1+2 variables. Following this, we use the Bianchi and Ricci identities, together with the Einstein equations, to derive a closed system of first-order propagation and constraint equations. Using the framework and outline for the 1+1+2 covariant approach in [1-3], we obtained the 1+1+2 equations for two fluids:

$$P_{1,\rho} = -{P_1}^2 + P_1\left[\mathbb{M}_1 - 3\mathcal{K} + \frac{7}{4}\right] + \mathbb{M}_1\left(\frac{1}{4} - \mathcal{K}\right) - P_1(P_2 - 2\mathbb{M}_2) - \mathbb{M}_1 P_2$$

$$P_{tot,\rho} = -{P_{tot}}^2 + P_{tot}\left[\mathbb{M}_{tot} - 3\mathcal{K} + \frac{7}{4}\right] + \mathbb{M}_{tot}\left(\frac{1}{4} - \mathcal{K}\right)$$

$$P_{2,\rho} = -{P_2}^2 + P_2\left[\mathbb{M}_2 - 3\mathcal{K} + \frac{7}{4}\right] + \mathbb{M}_2\left(\frac{1}{4} - \mathcal{K}\right) - P_2(P_1 - 2\mathbb{M}_1) - \mathbb{M}_2 P_1$$

$$\mathcal{K}_{,\rho} = 2\mathcal{K}\left(\frac{1}{4} - \mathcal{K} + \mathbb{M}_1 + \mathbb{M}_2\right)$$

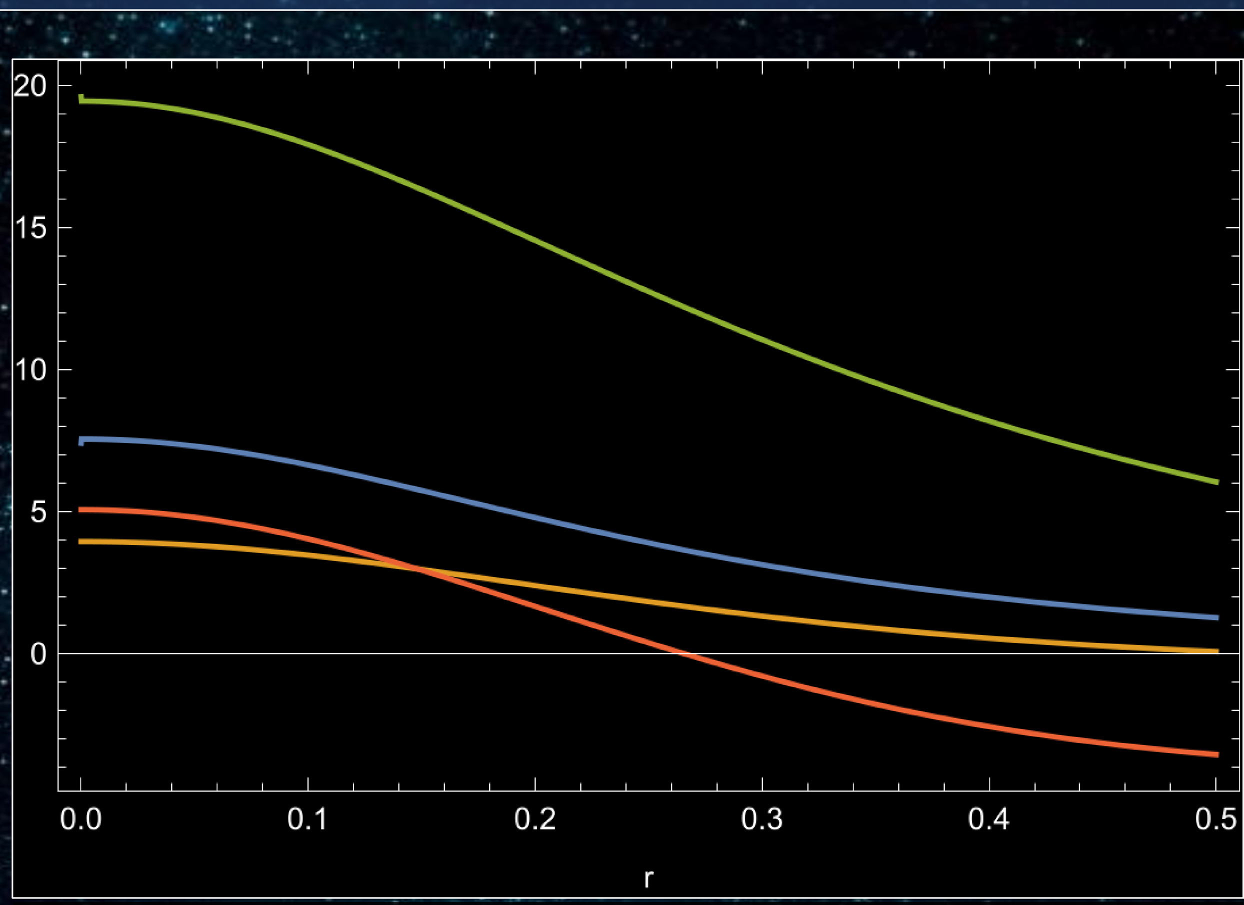

Figure 1: Energy density of fluid 1 (blue) and fluid 2 (green), as well as the pressure for fluid 1 (orange) and for fluid 2 (red), vs r for the Heintzmann IIa-Tolman IV object. We used the parameter values A = 0.5, R = 4, a = 3, and c = 1.

Solution reconstruction: Heintzmann IIa-Tolman IV relativistic object

We expand on the technique in [6,7] to obtain new two-fluid solutions from well-known single fluid ones. We obtained 3 new exact solutions. These 3 solutions presented a shell structure, i.e. the pressures of both fluids did not approach zero at the same value of r. In one example, we start from the Heintzmann IIa geometry and choose one of the fluids to be the Tolman IV fluid. In this way the energy density and the pressure of the two fluids will be given by

$$\mu_1 = \frac{z_2 z_3}{z_7^2}\left[-R^4\left(A^2(z_4+2) + z_4 r^2\right) - 4A^2r^4(z_4+1) + R^2\left(A^2r^2(5z_4+12) + A^4(z_4+2) + 4r^4(z_4+1)\right) - A^4r^2z_4 - 3r^6z_4\right]$$

$$\mu_2 = z_3\left(-z_5 + \frac{z_2}{z_7^2(r+R)^2}\times\left[3z_7r^2(z_4+4) - R^2z_7(z_4+2) + A^2\left(z_7(z_4+2) + 4r^2R^2 - 10r^4\right) - 2A^4r^2 - 2r^2\left(-5r^2R^2 + 6r^4 + R^4\right)\right]\right)$$

$$p_1 = \frac{z_8 z_3}{z_7}$$

$$p_2 = \frac{z_3(z_6 z_7 - z_8)}{z_7}$$

$$\begin{aligned}
z_0 &= \sqrt{4ar^2+1}\\
z_1 &= \sqrt{3-\mu_1 r^2}\\
z_2 &= \frac{1+ar^2}{3ar^2}\\
z_3 &= 1 - \frac{3\left(z_0+c\right)}{2z_2z_0}\\
z_4 &= \frac{ar^2\left\{ar^2\left[4az_0r^2 + 9(z_0-c)\right] + 18z_0\right\} + 4z_0}{z_0\left(ar^2+1\right)\left[ar^2\left(3cz_0+4ar^2-7\right)-2\right]}\\
z_5 &= \frac{3a\left[c\left(9ar^2+3\right) + \left(ar^2+3\right)z_0^3\right]}{z_0^2\left(ar^2+1\right)\left[ar^2\left(z_0+3c\right)-2z_0\right]}\\
z_6 &= \frac{3a\left[7acr^2 + 3\left(ar^2-1\right)z_0 + c\right]}{\left(ar^2+1\right)\left[ar^2\left(z_0+3c\right)-2z_0\right]}\\
z_7 &= \left(A^2+r^2\right)\left(r^2-R^2\right)\\
z_8 &= A^2 + 3r^2 - R^2
\end{aligned}$$

Figure 1 shows the pressures and energy densities of both fluids. We observe that although the matter variables are well-behaved, i.e. positive and decreasing, the pressure of fluid 2 (red) approaches zero before that of fluid 1 (orange). This implies that the solution represents a shell in an object's structure and must be matched smoothly with another shell before matching with an exterior solution, such as the Schwarzschild or Vaidya solutions in order to be complete. Figure 2 shows the square of the sound speed of fluid 2, for the same set of parameters.

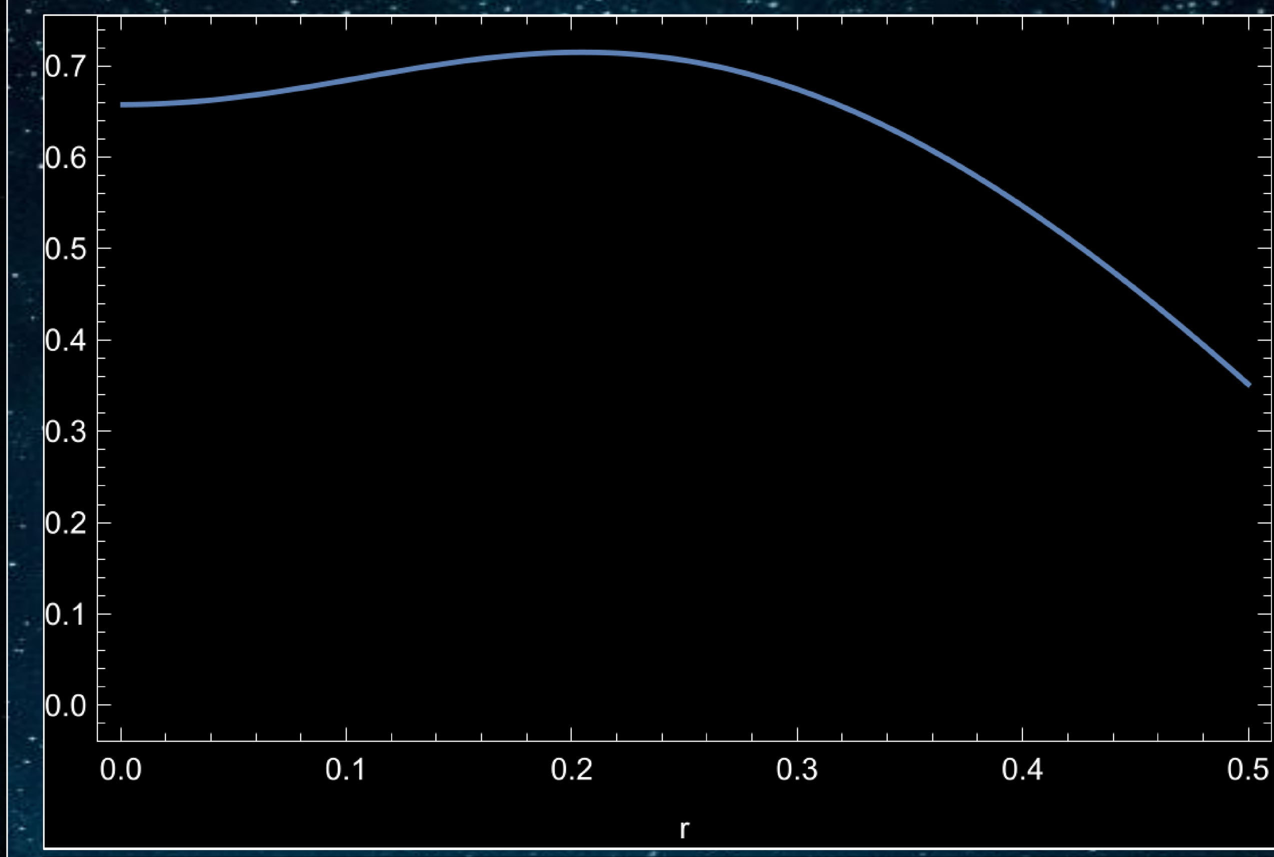

Figure 2: Plot of the square of the sound speed (∂p/∂μ) for fluid 2 for the Heintzmann IIa-Tolman IV object. We used the parameter values A = 0.5, R = 4, a = 3, and c = 1 as in Fig. 1.

A generating theorem

Boonserm et al. [4,5] developed transformation theorems able to map one "perfect fluid sphere" into another called generating theorems. These perfect fluid spheres are static, isotropic, spherically symmetric fluid distributions in curved spacetimes, which can then be associated with relativistic objects. In [6,7] it was shown that in the context of the covariant version of the TOV equations, the generating theorems assume the simple forms of linear solution deformation. We found that one of such theorems can also be used to obtain two-fluid solutions from known single-fluid ones. Given a solution of the TOV equations (1), we perform a deformation (2), where the subscript 0 represents the known solution, and tilde represents a deformed quantity. This corresponds to a transformation in the metric coefficients (3). Suppose that the starting solution is the Tolman IV metric. We were able to obtain a new solution, which is sourced by a first fluid, whose pressure is given by (4), where $\mathfrak{c}_1$ is a constant, *E* is the complete elliptical integral, and *F* is the elliptical integral of the first kind. The corresponding energy density is given by (5), and the pressure and energy density of the second fluid is given by (6) and (7) respectively, we have used (8). The metric coefficients, which combined with the matter variables represent the solution, are given explicitly in (9). Figure 3 shows the pressures and energy densities of both fluids. We observe that the pressure of the Tolman IV fluid (orange) approaches zero before the pressure of the deformed fluid (blue). Thus, our solution represents a shell in an object's structure and must be matched smoothly to another shell before being matched to an exterior solution. Figure 4 shows the square of the sound speed for both fluids, with the same set of parameter values as Figure 3.

$$\begin{aligned}
P_{,\rho} &= -P^2 + P\left[\mathbb{M} + 1 - 3\left(\mathcal{K} - \frac{1}{4}\right)\right] - \left(\mathcal{K} - \frac{1}{4}\right)\mathbb{M}\\
\mathcal{K}_{,\rho} &= -2\mathcal{K}\left(\mathcal{K} - \frac{1}{4} - \mathbb{M}\right)
\end{aligned} \tag{1}$$

$$\begin{aligned}
P_0 &\to P_0 + \tilde{P}\\
Y_0 &\to Y_0 + \tilde{Y}\\
\mathbb{M}_0 &\to \mathbb{M}_0\\
\mathcal{K}_0 &\to \mathcal{K}_0
\end{aligned} \tag{2}$$

$$\begin{aligned}
k_1 &\to k_1 \exp\left(\int \tilde{Y}\, d\rho\right)\\
k_2 &\to k_2\\
k_3 &\to k_3
\end{aligned} \tag{3}$$

$$\tilde{p}(r) = \left\{\left[2z_2z_3 + \mathfrak{c}_1\left(A^2+R^2\right)\right]\left(A^2+r^2\right) + 2z_1z_4\sqrt{A^2+r^2}\right\}^{-1}\times \frac{4z_1\sqrt{A^2+r^2}\left(A^2+R^2\right)}{z_4R^2} \tag{4}$$

$$\tilde{\mu}(r) = \left\{\left(2z_4\sqrt{A^2+r^2} + \mathfrak{c}_1 z_1\left(A^2+R^2\right) - 2z_1z_2z_3\right)\right\}^{-1}\times \frac{4z_3^2\sqrt{A^2+r^2}\left(A^2+R^2\right)}{R^2z_4^3} \tag{5}$$

$$p_0 = p_T = \frac{R^2 - A^2 - 3r^2}{R^2\left(A^2+2r^2\right)} \tag{6}$$

$$\bar{\mu}_0 = \frac{R^2\left(3A^2+2r^2\right) + 7A^2r^2 + 3A^4 + 6r^4}{R^2\left(A^2+2r^2\right)^2} - \frac{4z_3^2\sqrt{A^2+r^2}\left(A^2+R^2\right)}{R^2z_4^3}\times \left\{\left(2z_4\sqrt{A^2+r^2} + \mathfrak{c}_1z_1\left(A^2+R^2\right) - 2z_1z_2z_3\right)\right\}^{-1} \tag{7}$$

$$\begin{aligned}
z_1 &= \sqrt{R^2-r^2}\\
z_2 &= E\left(\sin^{-1}\left(\sqrt{\frac{R^2-r^2}{A^2+R^2}}\right), \frac{A^2}{A^2+2R^2}+1\right),\\
&\quad - F\left(\sin^{-1}\left(\sqrt{\frac{R^2-r^2}{A^2+R^2}}\right), \frac{A^2}{A^2+2R^2}+1\right)\\
z_3 &= \sqrt{A^2+2R^2}\\
z_4 &= \sqrt{A^2+2r^2}
\end{aligned} \tag{8}$$

$$\begin{aligned}
k_1 &= \left[\frac{2\mathfrak{c}_1}{A^2+R^2}\left(\frac{z_1z_4}{\sqrt{A^2+r^2}} - z_2z_3\right) + \sqrt{A^2+r^2} + \mathfrak{c}_2\right]^2\\
k_2 &= \frac{R^2{z_4}^2}{{z_1}^2\left(A^2+r^2\right)}\\
k_3 &= r^2
\end{aligned} \tag{9}$$

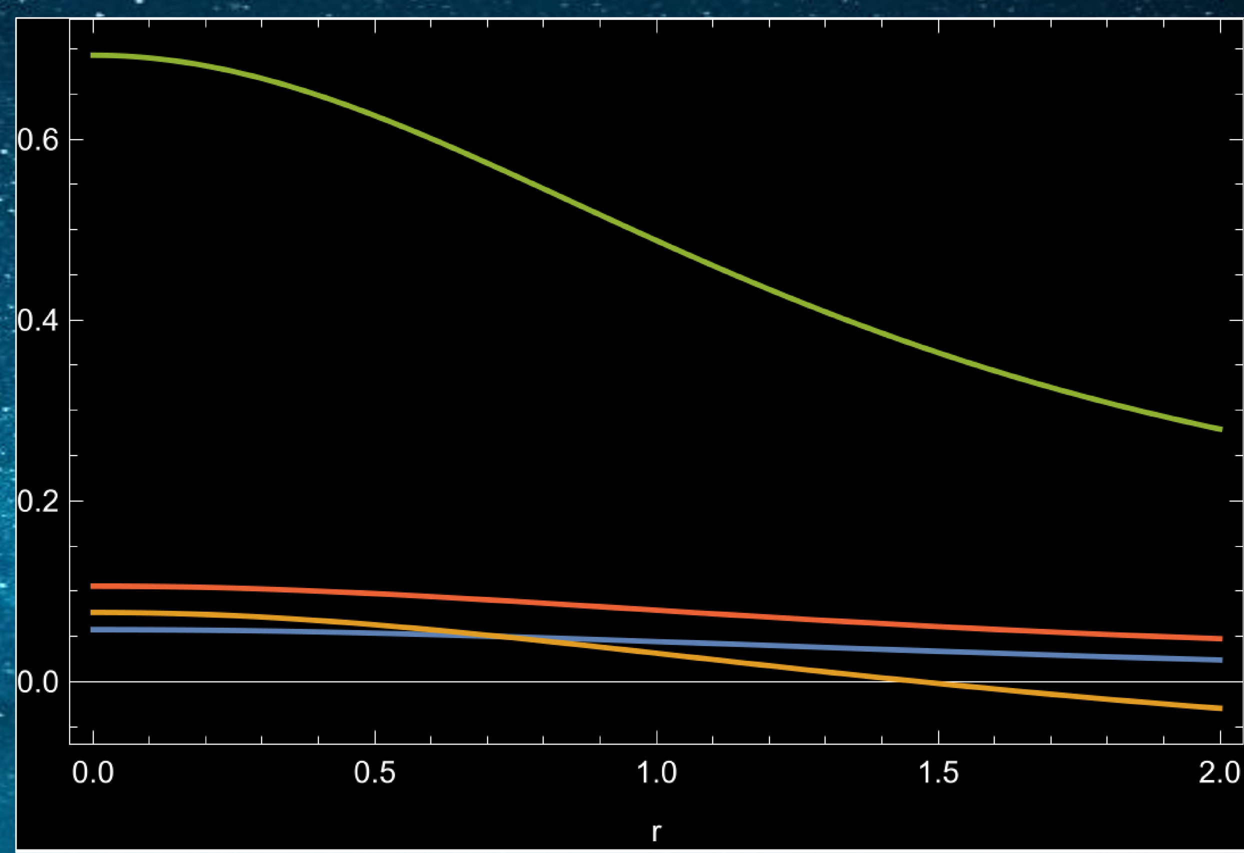

Figure 3: Pressure p_0 (orange), deformation pressure (blue), energy density μ_0 (green), and deformation energy density (red) vs r for the solution generated, with constant values A = −2.55, R = −3.6, and $\mathfrak{c}_1$ = −3.6.

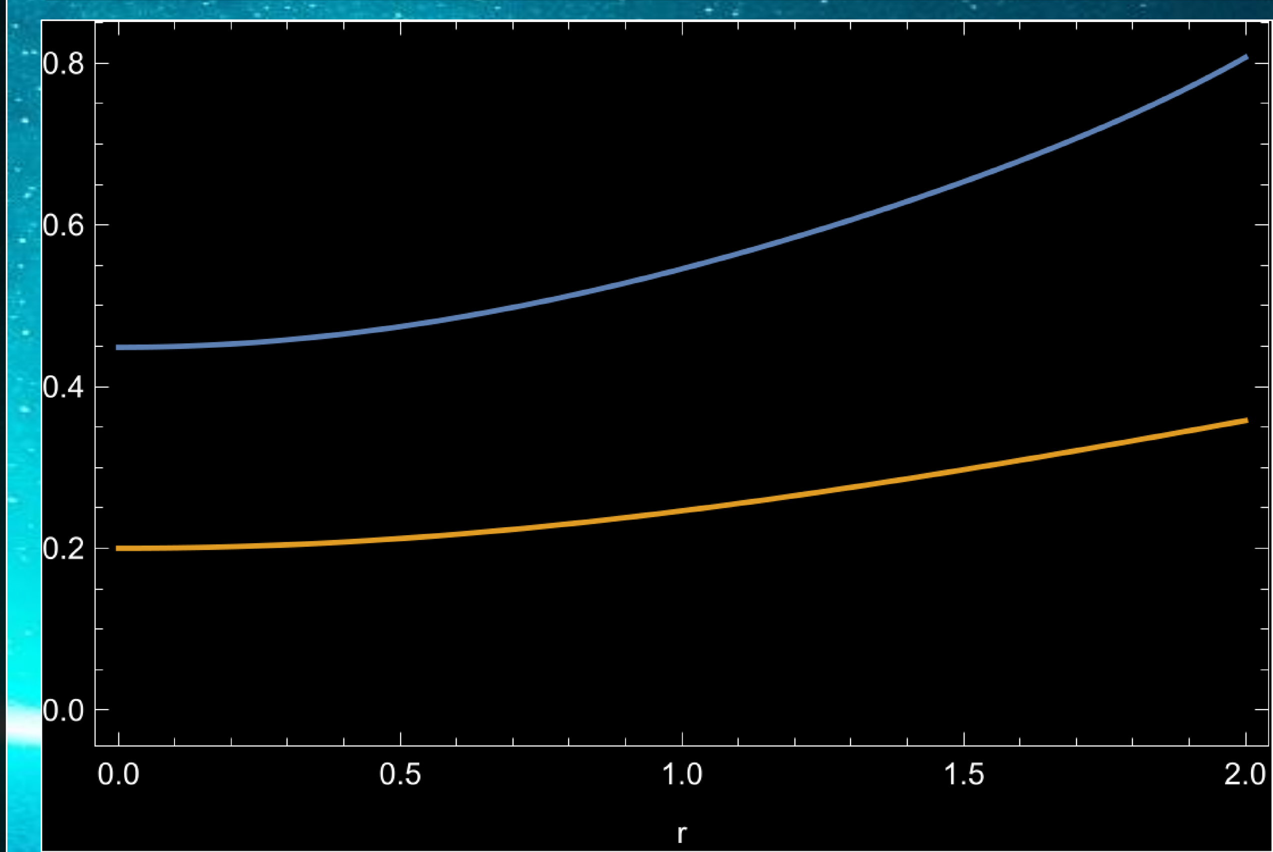

Figure 4: Square of the sound speed (∂p/∂μ) for the fluid with pressure p_0 (orange) and with deformed pressure (blue) vs r for the solution generated , with constant values A = −2.55, R = −3.6, and $\mathfrak{c}_1$ = −3.6.

Conclusion

We have presented a complete set of equations able to describe the interior solution of relativistic objects with more than one fluid source. These equations have been written by means of the 1+1+2 covariant formalism, which allows a relatively straightforward treatment of the many features of these kinds of systems. The 1+1+2 equations can be combined to obtain the covariant equivalent of the TOV equations. The properties of the TOV equations, however, can be more clearly appreciated when they are recast in a form that either contains dimensionless variables, or written in terms of quantities which are invariant under homological transformation. Using the 1+1+2 potential, we have defined variables with similar properties, which allow one to write the TOV equations as a closed system of the Riccati and Bernoulli equations, when the equation of state of matter is included. As in the case of the single-fluid solutions, these equations can be solved exactly with several techniques, other than direct resolution. We obtained a two-fluid generalization of the interior Schwarzschild solution. We formulated a reconstruction algorithm for the two-fluid TOV equations and obtained three new exact two-fluid solutions that are physically viable. One of the generating theorems has been employed to construct a new exact two-fluid solution from a known one. The new solution, obtained from the Tolman IV single fluid solution, represents a two-fluid relativistic star, comprising a perfect fluid and a fluid with a nontrivial equation of state.

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